

THE PRINCIPLE OF LEAST ACTION FOR FIELDS CONTAINING HIGHER ORDER DERIVATIVES⁺

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Abstract. *When generalizing the principle of least action for fields containing higher order derivatives, in general, it is not possible not to take into account the surface integrated term since it gives direct contribution to the forms of the equations of motion, of the energy-momentum tensor and of the angular-momentum tensor. This result is applied to two examples. In two dimensional gravity, it is essential to supplement Dirichlet condition for the surface integrated terms to vanish. On the contrary, in Hamiltonian description of the open rigid string, the equation of motion is modified by surface integrated terms. Boundary conditions are classified according to forms of certain equations of motion*

I. INTRODUCTION

Dynamical systems described in terms of higher-order Lagrangians have exhibited a lot of interesting aspects in connection with the gauge theories [1], gravity [2, 3], supersymmetry [4, 5], string models [6, 7], and other problems [8, 9, 10]. However, up to now it appears difficult to generalize the principle of least action for fields containing higher order derivatives due to the existence of surface integrated components [11, 12, 13].

The purpose of this paper is to apply different boundary conditions to consider the role of surface integrated terms, find the correct forms of the equations of motion, the energy-momentum tensor and the angular-momentum tensor in the appearance of boundary conditions.

The paper is organized as follows. Section II presents the derivation of energy-momentum tensor, angular-momentum tensor and study the influence of boundary conditions on Noether theorem. Section III is devoted to 2D gravity. In section IV, the corresponding formulae in the case of string theory are given. Section V is for the drawn conclusion.

II. FIELD SYSTEM AND BOUNDARY CONDITIONS

II.1. The principle of least action

Let us consider Lagrangian density for a free scalar field, which contains second order dervivatives

$$L(\varphi, \partial_\mu \varphi, \partial_\mu \partial_\nu \varphi). \quad (1)$$

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The equation of motion is derived by making an infinitesimal variation $\delta\varphi(x)$, and requiring the corresponding variation of the action to vanish

$$0 = \delta \int_{\Omega} L d^4x. \quad (2)$$

By denoting

$$L_{\varphi} = \frac{\partial L}{\partial \varphi} - \partial_{\mu} \frac{\partial L}{\partial \varphi_{,\mu}} + \partial_{\nu} \partial_{\mu} \frac{\partial L}{\partial \varphi_{,\mu\nu}}, \quad A_{\mu} = \left(\partial_{\nu} \frac{\partial L}{\partial \varphi_{,\mu\nu}} - \frac{\partial L}{\partial \varphi_{,\mu}} \right) \bar{\delta} \varphi - \frac{\partial L}{\partial \varphi_{,\mu\nu}} \bar{\delta} \varphi_{,\nu},$$

and integrating by parts (2), we can rewrite the principle of least action in the form

$$0 = \int_{\Omega} d^4x L_{\varphi} + \int_{\partial\Omega} (-A_{\mu} + L \delta x_{\mu}) d\sigma_{\mu}. \quad (3)$$

Ignoring surface integrated term in (3), the corresponding Euler-Lagrange equation will be

$$L_{\varphi} = 0 \quad \text{or} \quad \frac{\partial L}{\partial \varphi} - \partial_{\mu} \frac{\partial L}{\partial \varphi_{,\mu}} + \partial_{\nu} \partial_{\mu} \frac{\partial L}{\partial \varphi_{,\mu\nu}} = 0. \quad (4)$$

Let consider a space and time translation in which the variation of coordinate is $\delta x_{\mu} = \varepsilon_{\mu}$ and the field function transformation is

$$\varphi(x_{\mu}) \rightarrow \varphi'(x_{\mu}) = \varphi(x_{\mu}) - \varepsilon_{\mu} \partial_{\mu} \varphi. \quad (5)$$

The substitution of relation (5), (4) into expression (3) yields to the following local conservation law

$$T_{\rho\mu} = -\frac{\partial L}{\partial \varphi_{,\mu}} \varphi_{,\rho} - \frac{\partial L}{\partial \varphi_{,\mu\nu}} \varphi_{,\rho\nu} + \partial_{\nu} \frac{\partial L}{\partial \varphi_{,\mu\nu}} \varphi_{,\rho} + L \delta_{\mu\rho}, \quad \partial_{\mu} T_{\rho\mu} = 0. \quad (6)$$

$T_{\rho\mu}$ is the energy-momentum tensor of the system whose Lagrangian contains second order derivatives. From its local conservation we get four constants of motion

$$P_{\nu} = - \int d^3x T_{\nu 0}. \quad (7)$$

The last case we will consider is the invariance with respect to Lorentz transformations

$$x_{\mu} \rightarrow x'_{\mu} = x_{\mu} + \varepsilon_{\mu\nu} x_{\nu}, \quad (8)$$

$$\psi_{\alpha}(x) \rightarrow \psi'_{\alpha}(x') = \psi_{\alpha}(x) + \frac{1}{2} \varepsilon_{\mu\nu} (\Sigma_{\mu\nu})_{\alpha\beta} \psi_{\beta}(x). \quad (9)$$

Using (8), (9), (4) we find that the expression (3) gives

$$\frac{1}{2} \varepsilon_{\mu\rho} \int_{\partial\Omega} M_{\mu\nu\rho} d\sigma_{\mu} = 0, \quad (10)$$

where

$$\begin{aligned} M_{\mu\rho\sigma} = & x_{\mu} \left(T_{\rho\sigma} - \psi_{\alpha,\rho} \delta_{\rho\nu} \frac{\partial L}{\partial \psi_{\alpha,\sigma\nu}} \right) - x_{\rho} \left(T_{\mu\sigma} - \psi_{\alpha,\mu} \delta_{\rho\nu} \frac{\partial L}{\partial \psi_{\alpha,\sigma\nu}} \right) + \\ & + \left(\frac{\partial L}{\partial \psi_{\alpha,\sigma}} - \partial_{\nu} \frac{\partial L}{\partial \psi_{\alpha,\sigma\nu}} \right) (\Sigma_{\mu\nu})_{\alpha\beta} \psi_{\beta} - \frac{\partial L}{\partial \psi_{\alpha,\sigma\nu}} (\Sigma_{\mu\rho})_{\alpha\beta} \psi_{\beta,\nu}. \end{aligned} \quad (11)$$

It is clear that $M_{\mu\rho} = - \int_{\partial\Omega} M_{\mu\nu} d^3x$ is a 2nd rank tensor. This quantity is the angular-momentum tensor of the field. These results (6), (11) are agreed with the formulae in [14] if we put $n = 2$.

II.2. Boundary conditions

In the above subsection, we see that conservation laws associated with energy-momentum tensor and angular-momentum tensor (Noether theorem) is only correct if field function φ satisfies the equations of motion. This happens when the surface integrated term in (3) is vanished. Nevertheless, in the case of Lagrangian with second order derivatives, surface integrated components do not vanish naturally. Thus, to Noether theorem is satisfied, certain boundary conditions are necessary.

Now, we shall discuss more carefully about one of the conservation laws of Noether theorem - the energy conservation. When Lagrangian L contains second order derivatives, there are two independent configuration-space-type variables φ , $\dot{\varphi}$ and the corresponding canonical momenta

$$\pi = \frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial}{\partial t} \frac{\partial L}{\partial (\ddot{\varphi})} - \partial_i \frac{\partial L}{\partial (\partial_i \dot{\varphi})}, \quad s = \frac{\partial L}{\partial (\ddot{\varphi})} \quad (i = 1, 2, 3). \quad (12)$$

Substituting (12) into (7), the conserved energy P_0 will be

$$P_0 = \int d^3x (\pi \dot{\varphi} + s \ddot{\varphi} - L) + \frac{1}{2} \int d^3x \partial_i \left(\dot{\varphi} \frac{\partial L}{\partial (\partial_i \dot{\varphi})} \right). \quad (13)$$

In other words, Hamilton function of the system is

$$H = \int d^3x (\pi \dot{\varphi} + s \ddot{\varphi} - L). \quad (14)$$

Therefore, from (13), (14) we see that, unlike in the case of field containing first order derivatives, when the second order appears, Hamilton function H differs from conserved energy P_0 a surface integrated term

$$M = \int_R K d^3x = \frac{1}{2} \int_R \partial_i \left(\dot{\varphi} \frac{\partial L}{\partial (\partial_i \dot{\varphi})} \right) d^3x. \quad (15)$$

This quantity does not vanish in general, but depends on the boundary conditions. Its contribution to the energy formula will be considered more below.

In conclusion, since the surface integrated terms must be taken account, we must supplement boundary conditions for Noether theorem to be correct with higher order field. Upon the choice of boundary conditions, some changes will appear in the theorem. This will be discussed in the next two sections.

III. TWO DIMENSIONAL GRAVITY

We have shown that with high order Lagrangian, the contribution of surface integrated terms depends on boundary conditions. In this section, an appropriate boundary condition for these terms in 2D gravity to vanish will be derived.

In Teitlboim model [15], 2D gravity has Lagrangian

$$L = \sqrt{-g} k^{-1} (R - \Lambda). \quad (16)$$

By using $\|g_{\mu\nu}\| = e^\varphi \begin{pmatrix} (\eta^1)^2 - (\eta^\perp)^2 & \eta^1 \\ \eta^1 & 1 \end{pmatrix}$, the above Lagrangian can be rewritten in the form

$$L = \frac{1}{\eta^\perp} \left(\dot{\varphi} - \varphi' \eta - 2\eta^{1'} \right)^2 - \eta^\perp \varphi'^2 + 4\eta^\perp \varphi'' + 2\eta^\perp \Lambda e^\varphi, \quad (17)$$

where only scale conformal factor φ is dynamical variable ($\varphi(t, x)$, $x_\mu = (x_0, x_1) = (t, x)$, $0 < x < L$, $-\infty < x < \infty$), other conformal constant components of the metric η^1 , η^\perp are unchanged external field. From (4), we get the Euler-Lagrange equation

$$L_\varphi = \eta^\perp \Lambda e^\varphi - \partial_0 \left[(\eta^\perp)^{-1} \Omega \right] - \partial_1 \left[-(\eta^\perp)^{-1} \Omega \eta^1 - \eta^\perp \varphi' - 2\eta^{1'} \right] = 0, \quad (18)$$

in which $\Omega = \dot{\varphi} - \varphi' \eta^1 - 2\eta^{1'}$.

Now, let us consider Dirichlet boundary conditions

$$\begin{aligned} \delta\varphi(t, x=0) &= \delta\varphi(t, x=L) = 0, \\ \delta\varphi'(t, x=0) &= \delta\varphi'(t, x=L) = 0. \end{aligned}$$

With these conditions, one part of the surface integrated term in (3) vanishes. Thus, for the equation of motion is satisfied, since field function does not disappear at time boundary, it is essential to supplement auxiliary conditions

$$C(t, x) = \frac{\partial L}{\partial \dot{\varphi}} - \partial_i \frac{\partial L}{\partial (\partial_i \dot{\varphi})} = 0, \quad D(t, x) = \frac{\partial L}{\partial \dot{\varphi}} = 0. \quad (19)$$

This is also an extension of Neuman boundary conditions.

It is obvious that energy-momentum tensor (6) is modified by relations (19). Its new formulae in this case will be

$$T'_{\rho 0} = -\frac{\partial L}{\partial \dot{\varphi}} \varphi_{,\rho} - \frac{\partial L}{\partial \dot{\varphi}_{,\nu}} \varphi_{,\rho\nu} + \partial_0 \frac{\partial L}{\partial \dot{\varphi}_{,\nu}} \varphi_{,\rho} + L \delta_{\mu\rho}, \quad T'_{\rho 1} = L \delta_{1\rho}. \quad (20)$$

Using extended Neuman conditions (19), we also have the new surface integrated term (15)

$$M = \frac{1}{2} \int_R \partial_1 \left(\dot{\varphi} \frac{\partial L}{\partial (\partial_i \dot{\varphi})} \right) dx = \dot{\varphi} \frac{\partial L}{\partial \dot{\varphi}} \Big|_{x=0}^{x=L}. \quad (21)$$

Since Lagrangian (17) does not depend explicitly on $\dot{\varphi}'$, which means $\frac{\partial L}{\partial \dot{\varphi}'} = 0$, the contribution of surface integrated term (21) in energy formula is vanished.

In summary, in 2D gravity, using Dirichlet boundary conditions, it is necessary to add extended Neuman conditions for the equation of motion to be satisfied. Consequently, some minor changes appear in energy-momentum tensor and surface integrated components does not appear.

IV. STRING THEORY

In the example of 2D gravity, boundary condition of field function was considered. The result is that when Dirichlet conditions are applied, surface integrated terms vanish. In this section, for string theory, boundary conditions of space-time coordinates are studied.

The Lagrangian for the rigid string has the form

$$L = \sqrt{-g} (-\gamma + \alpha \Delta x^\mu \Delta x_\mu), \quad (22)$$

where $\alpha \neq 0$, $-\gamma > 0$ are constants, α is a dimensionless constant characterizing for the curvature of the word-sheet of the string. For $\alpha = 0$, we would obtain the usual Nambu-Goto string. Δ is Laplace-Beltrami operator.

Let consider this Lagrangian in two dimensional parametrization $x = (\tau, \sigma)$, $0 \leq \sigma \leq \pi$, $-\infty < \tau < \infty$. For this parametrization, in formulae in Section II, instead of making derivatives with respect to variable x , the variables now are parameters (τ, σ) . Moreover, derivatives with respect to φ are now transformed into derivatives with respect to x .

In the virtual of the above assumptions, form (4) and [12], we have the equation of motion

$$(\gamma - \alpha \Delta x^\mu \Delta x_\mu) \Delta x_\mu + 2\alpha \left(\Delta (\Delta x_\mu) - g^{i\rho} x_{,i}^\nu x_{\mu,j} \Delta (\Delta x_\nu) \right) - 4\alpha g^{i\rho} g^{kz} (\Delta x_\nu)_{,j} x_{,k}^\nu \nabla_i x_{\mu,z} = 0, \quad (23)$$

where $\nabla_i x_{\mu,z}$ denotes the covariant derivative of the covariant vector $x_{\mu,z}$; $z = 0, 1$, defined with the use of Christoffel symbols for the metric. Equation (23) is rather complicated since it contains fourth order derivatives and nonlinear components. For $\alpha = 0$, this equation will yield to Nambu-Goto equation $\Delta x_\mu = 0$.

In the case of open rigid string, we choose boundary conditions as in [12]

$$\delta x_\mu(\tau, \sigma) = 0 (NG), \quad \delta x_{\mu,0}(\tau, \sigma) = 0 \text{ for } \tau = \tau_1, \tau_2, \sigma \in [0, \pi]. \quad (24)$$

Then, for the equation of motion (23) is satisfied, the auxiliary conditions which the field function need to obey are (the procedure is the same as in section III)

$$\begin{aligned} B_\mu(\tau, \sigma = 0) &= 0, \quad B_\mu(\tau, \sigma = \pi) = 0, \\ C_\mu(\tau, \sigma = 0) &= 0, \quad C_\mu(\tau, \sigma = \pi) = 0, \end{aligned} \quad (25)$$

where

$$\begin{aligned} B_\mu(\tau, \sigma) &= \sqrt{-g} (\gamma - \alpha \Delta x^\mu \Delta x_\mu) g^{1i} x_{\mu,i} + 2\alpha \sqrt{-g} g^{jk} x_{,i}^\lambda x_{\lambda,jk} \Delta x_\mu + \\ &+ 4\alpha \sqrt{-g} \Delta x_\sigma x_{,ij}^\sigma g^{1j} g^{ik} x_{\mu,k} + 2\alpha \partial_0 (\sqrt{-g} g^{\dot{0}1} \Delta x_\mu) + \\ &+ 2\alpha \partial_j (\sqrt{-g} g^{1j} \Delta x_\mu), \end{aligned} \quad (26)$$

and

$$C_\mu(\tau, \sigma) = 2\alpha \sqrt{-g} g^{11} \Delta x_\mu. \quad (27)$$

The energy density momentum p_μ corresponding to Lagrangian (22) has the form

$$p_\mu = \sqrt{-g} g^{0j} (\gamma - \alpha \Delta x_\sigma \Delta x^\sigma) x_{\mu,j} + 2\alpha \partial_0 (\sqrt{-g} g^{00} \Delta x_\mu) + 2\alpha \sqrt{-g} g^{0i} g^{jk} (2\Delta x_\sigma x_{,i\rho}^\sigma x_{\mu,k} + x_{,jk}^\lambda x_{\lambda,i} \Delta x_\mu). \quad (28)$$

From (13), (14), (15), we obtain the rate at which the energy change with respect to the change in time [12]

$$\frac{dH}{dt} = \text{the boundary terms} = -\partial_0 \left(\int_0^\pi d\sigma \partial_1 \left(\dot{\vec{x}} \frac{\partial L}{\partial \dot{\vec{x}}} \right) \right). \quad (29)$$

In general, the r.h.s of equation (29) does not vanish. Thus, conserved energy derived from energy-momentum tensor differs from Hamilton function a surface integrated term. This also means the evolution with time of Hamilton function of the rigid string is not equal to its conserved energy

V. CONCLUSION

Classical equations of motion, energy-momentum tensor and angular - momentum tensor of higher order derivative Lagrangian are systemized and red-erived in the appearance of boundary condition. We have also pointed out that whether surface integrated terms can be ignored or not depends on the boundary conditions. Two kinds of boundary conditions are considered. In 2D gravity, we use Dirichlet conditions of field function for those terms to vanish. On the contrary, in open rigid string, when boundary conditions of space-time coordinates are chosen, it is unable to ignore surface integrated terms. Moreover, some changes appear in the time-components of energy-momentum tensor.

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